SHOCK WAVES IN MAGNETO-GASDYNAMICS

(UDARNYE VOLNY V MAGNITHOI GAZODINAMIKE)

PMM Vol.23, No.3, 1959, pp.557-563

M. N. KOGAN
(Moscow)

(Received 7 February 1959)

In this work the nature of the shock polar is investigated for various parameters of the stream and of the magnetic field.

1. Statement of the problem. Shock waves are considered in an ideal gas with inifinite electrical conductivity. Let $\mathbf{V}_{1,2}$, $\mathbf{H}_{1,2}$, $p_{1,2}$, $\rho_{1,2}$ and $T_{1,2}$ be respectively the velocity and magnetic field vectors, pressure, density and temperature ahead of (subscript 1) and behind (subscript 2) a shock wave. We will assume that $\mathbf{V}_1$ and $\mathbf{H}_1$ are parallel. This assumption does not reduce the generality, because the case of arbitrary directions of $\mathbf{V}_1$ and $\mathbf{H}_1$ can always be reduced to that under consideration by appropriate choice of a moving inertial coordinate system [1]. We will consider the flow in the plane containing the coincident vectors $\mathbf{V}_1$ and $\mathbf{H}_2$ and the normal to the surface of the shock wave. Then the vectors $\mathbf{V}_2$ and $\mathbf{H}_2$ are also parallel and lie in the same plane, and the entire flow is plane (Fig. 1).

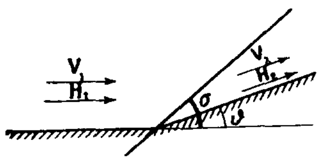

Fig. 1.

It is shown in the work [2] that magneto-hydrodynamic flows are determined by two dimensionless parameters: the Mach number M and the number N determined by the relation

$$N^2 = \frac{H^2/8\pi}{{}^1/_2\,\varkappa p} = \frac{H^2/4\pi\rho}{\varkappa p/\rho} = \frac{a_A{}^2}{a^2}$$

where a_A is the Alfvéen speed and a the ordinary speed of sound, and κ is the ratio of specific heats.

In the notation indicated above and in Fig. 1, the known relations [1] for a shock wave may be written in the form

$$\frac{H_2}{H_1}=\frac{\rho_2 V_2}{\rho_1 V_1}, \qquad \frac{V_2}{V_1}=\frac{N^2}{M^2}\frac{\sin\vartheta}{\sin(\sigma-\vartheta)\cos(\sigma-\vartheta)}+\frac{\cos\sigma}{\cos(\sigma-\vartheta)}$$

$$\frac{\rho_2}{\rho_1}=\frac{V_1}{V_2}\frac{\sin\sigma}{\sin(\sigma-\vartheta)}, \qquad \frac{p_2}{p_1}=\frac{\rho_2}{\rho_1}\left[1-\tfrac{1}{2}(\varkappa-1)M^2\left(\frac{V_2^2}{V_1^2}-1\right)\right] \tag{1.1}$$

$$\frac{p_2}{p_1}=1-\varkappa M^2\sin^2\sigma\left(\frac{\rho_1}{\rho_2}-1\right)-\tfrac{1}{2}\varkappa N^2\left[\frac{\sin^2\sigma\cos^2(\sigma-\vartheta)}{\sin^2(\sigma-\vartheta)}-\cos^2\sigma\right]$$

In the works [1] and [3] it is shown that in a real shock wave (with increasing entropy)

$$\frac{p_2}{p_1}>1, \qquad \frac{\rho_2}{\rho_1}>1, \qquad \frac{V_2}{V_1}<1$$

The problem discussed in the present work consists in elucidating the nature of changes in flow parameters behind the shock wave as they depend upon the parameters M and N ahead of the shock and the angle of inclination ϑ of the velocity and field vectors to the wave. In view of the complexity of the relations (1.1), in the analysis only special characteristic points of the shock polar are given, and certain degenerate regimes A complete analysis by calculation of the behavior of the functions near these characteristic points permits the general nature of the shock polar to be shown.

2. Weak shock waves. In the work [2] weak shock waves were studied, for which the variation of all flow and field parameters is proportional to the angle of deflection of the velocity vector. Such shock waves we will call weak shock waves of the first family. The angle of inclination of a shock wave of the first family of zero intensity is determined by the expression [2]

$$\operatorname{tg}\sigma_0=\pm\sqrt{\frac{M^2-N^2(1-M^2)}{(M^2-1)(M^2-N^2)}} \tag{2.1}$$

As is easily seen from (2.1), these waves exist for $N<1$ in the range $M_a\leqslant M\leqslant N$ and $1\leqslant M\leqslant\infty$, and for $N>1$ in the range $M_a\leqslant M\leqslant 1$ and $N\leqslant M\leqslant\infty$, where $M_a=N(1+N^2)^{-1/2}$

The angle by which a shock wave of non-zero intensity deviates from σ_0 is proportional to the deflection angle ϑ of the stream:

$$\delta=\sigma-\sigma_0=\frac{(3+\varkappa)(1-M^2)N^2+(\varkappa+1)(N^2-M^2)}{4[M^2-N^2(1-M^2)](1-M^2)}\vartheta \tag{2.2}$$

The pressure behind the shock wave is equal to

$$p_2=p_1+\rho V_0^2\frac{M^2-N^2}{M^4\sin\sigma_0\cos\sigma_0}\vartheta \tag{2.3}$$

For $M < 1$ the deflection of the stream through a positive angle ϑ at a weak shock wave of the first family can take place only for $\sigma_0 \geqslant 90^\circ$ (in this case $\cos\sigma_0 < 0$ and $p_2 > p_1$). According to (2.2), under the same conditions $\delta > 0$, consequently, as ϑ increases, the shock wave approaches the axis. As $N \to 0$ with $M > 1$ the wave under consideration approaches the weak wave of ordinary gas dynamics. In magneto-gasdynamics a new type of weak shock wave appears, which we will call a wave of the second family, and which has no analog in ordinary gasdynamics. These waves occur for $M^2 = N^2 + \Delta$, $\sigma = 90^\circ + \tau$ and $\vartheta = 2\tau + \epsilon$, where Δ and ϵ are small, and $0 \leqslant \tau \leqslant 90^\circ$. In this case, neglecting squares of small quantities in (1.1) we have

$$\frac{V_2}{V_1} = 1 + \varepsilon \operatorname{tg}\tau - \frac{\Delta}{M^2}, \qquad \frac{p_2}{p_1} = \varkappa M^2 \left(\cos^2\tau \frac{\Delta}{M^2} - \varepsilon \operatorname{tg}\tau\right)$$
$$\frac{\rho_2}{\rho_1} = 1 + \frac{\Delta}{M^2}, \qquad \varepsilon = -\left[\frac{1 - M^2}{M^2} + \varkappa \sin^2\tau\right]\frac{\Delta}{M^2} \qquad (2.4)$$

Hence, it is clear that we must have $\Delta > 0$; that is, the second type of shock wave exists for $M > N$.

For $M = N$ the stream is turned through any angle up to 180° without changes in the flow and field parameters*. This type of wave has a simple physical interpretation. Consider a wave of this type (Fig. 2). From the geometry of the flow it is clear that the angles μ_1 and μ_2 between the velocity and field vectors and the normal to the wave are equal to each

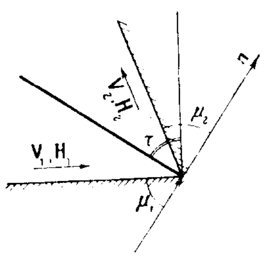

Fig. 2.

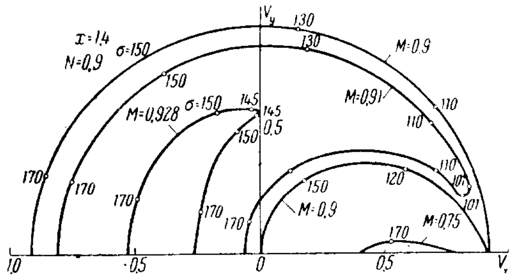

Fig. 3.

other and to τ; that is, $\mu_1 = \mu_2 = \mu = \tau$. Since $p_1 = p_2$, $\rho_1 = \rho_2$, $V_1 = V_2$ and $H_1 = H_2$, it is clear that the mass, normal momentum and energy of the stream are conserved. The change in tangential momentum is determined such that, in contrast to the ordinary shock wave, in the magneto-hydrodynamic wave a finite force acts on the fluid. In fact, since the current across any surface is equal to the integral of the field around the contour of that surface

* These waves are called rotatory shocks [1].

$$4\pi I = c \oint \mathbf{H} dl$$

and since the field suffers a discontinuity at the shock, a finite current flows in the shock. Since this current flows in a magnetic field, a finite force acts upon it, having in the general case normal as well as tengential components. A shock wave in an infinitely conducting gas is basically distinguished from an ordinary shock wave by the appearance of this force,

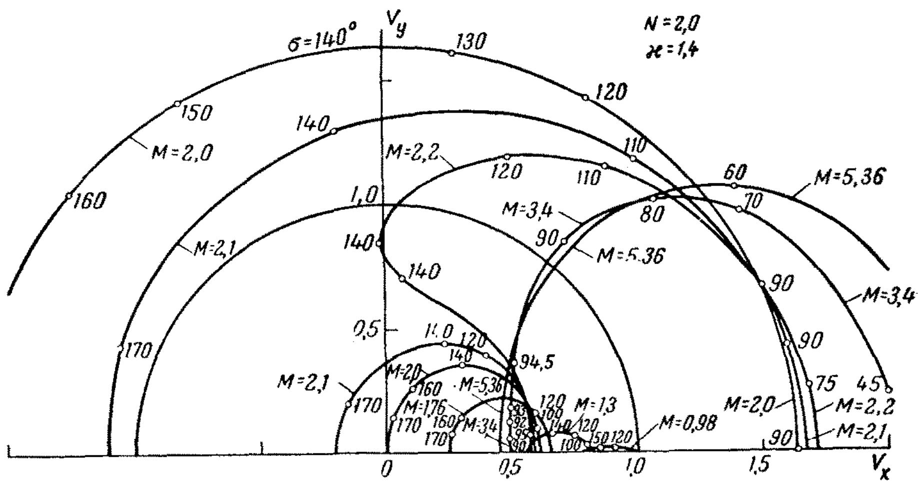

Fig. 4.

since the existence of a magnetic field and infinite conductivity affect neither the conservation of mass nor the conservation of energy*.

In the case considered the current per unit length of shock is equal to $1/2\,\pi^{-1}\,cH \sin\mu$. This current flows in the magnetic field $H\cos\mu$ normal to the shock, so that the force acting along the shock is equal to $1/2\,\pi^{-1}\,cH^2 \sin\mu\cos\mu$. This force must be equal to the change in tangential impulse

$$2\rho V^2 \sin\mu\cos\mu = {}^1/_2\,\pi^{-1} H^2 \sin\mu\cos\mu$$

This equality clearly applies to $M = N$, which was assumed.

We note that the last type of wave occurs also in an imcompressible fluid for $4\pi\rho_1 V_1^2 = H_1^2$. In the hodograph plane of the velocity (Figs. 3 and 4) the polar for the shock wave of the second family with $M = N$ is clearly represented by a circle.

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* The flux of the Poynting vector $\mathbf{s} = (4\pi)^{-1}\,c\,\mathbf{E}\times\mathbf{H}$ through a control surface enclosing both sides of the shock surface is equal to zero.

3. Normal shock wave ($\sigma = 1/2\pi$). If $\mathbf{V}_1$ and $\mathbf{H}_1$ are perpendicular to the shock front, then one of the possible flows is a flow with a continuous field. In this case there is no current in the shock and the shock wave is identical with the ordinary one. However, in magneto-gasdynamics a flow is possible with deflection of the velocity and field vectors. In this case the force acting on the current arising at the wave turns the stream.

Putting $\sigma = 1/2\pi$ in (1.1) we have

$$\frac{V_2}{V_1} = \frac{N^2}{M^2}\frac{1}{\cos\vartheta}, \qquad \frac{p_2}{p_1} = 1 + \varkappa(M^2 - N^2) - {}^1/_2\varkappa N^2 \operatorname{tg}^2\vartheta$$

$$\frac{\rho_2}{\rho_1} = \frac{M^2}{N^2}, \qquad \operatorname{tg}^2\vartheta = \frac{(N^2 - M^2)\,[2 + (\varkappa-1)\,M^2 - (\varkappa+1)\,N^2]}{N^4}$$

It is easy to verify that such shock waves of compression ($p_2/p_1 > 1$) exist only for $N > 1$ and $N \leqslant M \leqslant M_b$, where

$$M_b^2 = \frac{\varkappa+1}{\varkappa-1}\left(N^2 - \frac{2}{\varkappa+1}\right)$$

Thus in this range three regimes of flow are possible behind a normal shock (with $\vartheta > 0$, $\vartheta = 0$ and $\vartheta < 0$). The point $M = M_b$ is also special for the behavior of the shock wave near $\sigma = 1/2\pi$ and $\vartheta = 0$. In fact, with $\sigma = 1/2\pi - \delta$ and small we have

$$\delta = \frac{2 + (\varkappa-1)\,M^2 - (\varkappa+1)\,N^2}{2\,(M^2-1)}\,\vartheta$$

For $M > 1$ and $M < M_b$ the numerator on the right is negative. Hence, as ϑ increases σ also increases (that is, σ becomes $> 1/2\pi$). For $M > M_b$, on the other hand, as ϑ increases the angle of inclination σ decreases, as it does also along the shock polar of ordinary gasdynamics near the normal shock.

4. Shock waves with $M = N$. It was noted in Section 2 that for $N = M$ there exist weak shock waves of a second family at which the stream and the field change their direction by any angle without a change in the flow parameters. On the other hand, in Section 3 we saw that for $M > 1$ and any N the usual normal shock wave exists. This wave clearly does not belong to the class of weak shock waves of the second family, and consequently belongs to another branch of the shock polar. It is found that for $M = N$ there exists a class of shock waves whose inclination σ makes a right angle with the direction ϑ of the stream behind the shock. Putting $\sigma = 1/2\pi + \vartheta$ in (1.1) we obtain

$$\frac{p_2}{p_1} = \frac{V_1 \sin\sigma}{V_2}\left[1 - \tfrac{1}{2}(\varkappa-1)\,M^2\left(\frac{V_2^2}{V_1^2} - 1\right)\right], \qquad \frac{\rho_2}{\rho_1} = \left(\frac{V_1}{V_2}\right)\sin\sigma$$

$$\left(\frac{V_2}{V_1}\right)_{1,2} = \frac{[2+\varkappa M^2(1+\sin^2\sigma)] \pm \sqrt{R}}{2(\varkappa+1)M^2\sin\sigma} \tag{4.1}$$

$$R = [2+\varkappa M^2(1+\sin^2\sigma)]^2 - 4(\varkappa+1)M^2\sin^2\sigma\,[2+(\varkappa-1)M^2]$$

Here the indices 1 and 2 correspond to the upper and lower signs.

With $\sigma = 1/2\pi + \delta$, where δ is a small quantity and $M < 1$, we have

$$\left(\frac{V_2}{V_1}\right)_1 = \left[\frac{2+(\varkappa-1)M^2}{(\varkappa+1)M^2} + O(\delta^2)\right] > 1, \qquad \left(\frac{V_2}{V_1}\right)_2 = 1 - \frac{\delta^2}{2(1-M^2)} < 1$$

and for $M > 1$

$$\left(\frac{V_2}{V_1}\right)_1 = 1 + \frac{\delta^2}{2(M^2-1)} > 1, \qquad \left(\frac{V_2}{V_1}\right)_2 = \frac{2+(\varkappa-1)M^2}{(\varkappa+1)M^2}\left[1 - \frac{\delta^2}{2(M^2-1)}\right] < 1$$

Thus, only the second root corresponds to a real shock wave. For $\delta \to 1$ and $M > 1$ we have the usual normal shock wave, and for $M < 1$ a weak wave. Consequently for $M < 1$ and $M = N$ both branches of the shock polar start in one point, corresponding to a weak shock wave with $\vartheta = 0$. For $\sigma = \pi - \delta$

$$\left(\frac{V_2}{V_1}\right)_{1,2} = \frac{[2+\varkappa M^2(1+\delta^2)] \pm [2+\varkappa M^2(1+\delta^2)]}{(\varkappa+1)M^2\delta} \mp \frac{2+(\varkappa-1)M^2}{(2+\varkappa M^2)}\delta$$

Here again, only the second root corresponds to a real wave:

$$\left(\frac{V_2}{V_1}\right)_2 = \frac{2+(\varkappa-1)M^2}{2+\varkappa M^2}\delta < 1$$

We note that the magnitude V_2 of the velocity behind the wave tends to zero as $\sigma \to \pi$, and ϑ tends to $1/2\pi$.

The form of the branch for strong shock waves when $M = N$ is shown in Figs. 3 and 4. We note that with increasing angle of inclination of the wave (going from $\sigma = 1/2\pi$ to $\sigma = \pi$) the wave becomes continually weaker; that is, p_2/p_1 falls from the value $p_2/p_1 = (\kappa+1)^{-1}[2\kappa M^2 - (\kappa-1)]$ corresponding to the ordinary normal shock to the value $p_2/p_1 = 1/2(2 + \kappa M^2)$. However, the entropy loss increases (that is, the ratio $p_2/\rho_2^{\kappa}/p_1/\rho_1^{\kappa}$ falls).

5. Shock waves with $\sigma \to \pi$. It is evident that as $\sigma \to \pi$ two possibilities arise*: $\vartheta \to 0$ and $\vartheta \to \pi$. Let $\sigma = \pi - \delta$ and ϑ 0. Then

$$\frac{V_2}{V_1} = -\frac{N^2}{M^2}\frac{\vartheta}{\delta+\vartheta} + 1, \qquad \frac{p_2}{p_1} = 1 - \frac{\varkappa}{2}N^2\left[\frac{\delta^2}{(\delta+\vartheta)^2} - 1\right]$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2}\frac{\delta}{\delta+\vartheta}, \qquad \delta = \frac{1}{2}(-B \pm \sqrt{B^2-4AC})A^{-1}\vartheta \tag{5.1}$$

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* The case $M = N$ is excluded, when $\to 1/2\pi$ as $\sigma \to \pi$.

where

$$A = M^2 - N^2(1 - M^2), \quad B = {}^1/_2\{M^2N^2 - [4 + (x + 1)N^2](N^2 - M^2)\}$$

$$C = (1 + {}^1/_2 \varkappa N^2)(M - N^2)$$

For $\sigma = \pi - \delta$ and $\vartheta = -\theta$ we have

$$\frac{V_2}{V_1} = \frac{N^2}{M^2}\frac{\theta}{\theta - \delta} - 1, \qquad \frac{p_2}{p_1} = 1 - \frac{\varkappa}{2}N^2\left[\frac{\delta^2}{(\theta - \delta)^2} - 1\right]$$
$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2}\frac{\delta}{\theta - \delta}, \qquad \delta = \frac{1}{2}(B \pm \sqrt{B^2 - 4AC})A^{-1}\theta \tag{5.2}$$

It is easy to see that among the shock waves determined by the relations (5.1) only those will be real for which ϑ and δ are positive. Therefore, these waves can exist for $M_a \leqslant M \leqslant N$. Here only the root corresponding to the upper sign is meaningful. Analogously, waves turning the stream through almost 180°, determined by the relations (5.2), can exist only for $N \leqslant M \leqslant M_0$, where M_0 is the Mach number for which $B^2 - 4AB$ vanishes. For $M > M_0$ the square root in (5.1) and (5.2) becomes imaginary. Values of M_c for different N are given in Fig. 5, where the values M_a, $M = N$ and M_b are also given. We note that M_c is less than M_b almost everywhere. Thus, for $N \leqslant M \leqslant M_c$ these exist two waves turning the stream through an angle near to π.

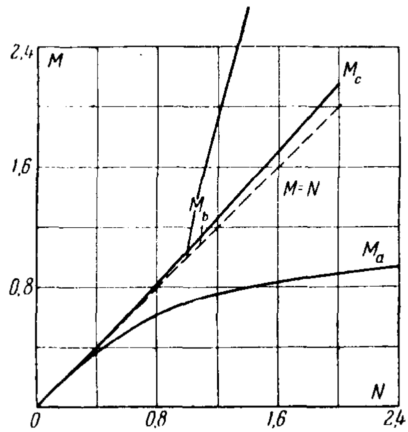

Fig. 5.

6. General nature of the shock polar. The analysis given above permits us to elucidate the general nature of the behavior of the shock polar for different values of the parameters M and N of the stream and the field. It is convenient to consider separately the cases $N < 1$ and $N > 1$.

The case $N < 1$ (Fig. 3). Here there is no shock wave for $M < M_a$. For $M_a \leqslant M < N$ there is one branch of the shock polar, to which correspond shock waves with an angle of inclination $\sigma \geqslant \sigma_0 \geqslant 1/2\pi$. The pressure behind the wave and the loss due to entropy rise increase in moving along the polar from $\sigma = \sigma_0$ to $\sigma = \pi$. For $\sigma = \sigma_0$ the weak shock of the first family exists. For $M = M_a$ all the polars tend to a point. As M increases, the point corresponding to $\sigma = \pi$ moves toward the origin of coordinates, which it reaches when $M = N$. For this value of M the weak shock waves of the second family appear (Section 2), whose polar is represented by a circle. Both branches of the polar start at the point $V_1 = V_2$ $\vartheta = 0$ and $\sigma = \sigma_0 = 1/2\pi$. With further increase in Mach number, the points of both branches corresponding to $\sigma = \pi$ and $\vartheta = \pi$ begin to approach each other, coinciding at $M = M_c$. As was observed in Section 2, for $1 > M > N$ there is no weak shock with $\vartheta \sim 0$. The juncture point of the two branches of the shock polar detaches itself from the axis and with increase in Mach number tends toward continually larger angles . For $M = M_c$ all the polars tend to a point. For $M_c < M < 1$ no shock wave exists. For $M > 1$ there exists one branch of shock waves qualitatively similar to the shock polar of ordinary gasdynamics.

The case $N > 1$ (Fig. 4). Here, just as in the preceding case, there is no shock wave for $M < M_a$. For $M_a \leqslant M < 1$ there is a branch of the polar corresponding to waves with an angle of inclination $\pi > \sigma > \sigma_0 > 1/2\pi$. For $M = 1$ the angle of inclination of the weak wave of the first family σ_0 becomes equal to 1/2 5. Here, properly speaking, the weak shock wave is at the same time also a normal shock. For $M > 1$ the shock polars begin with a normal shock ($\vartheta = 0$; that is, with the normal shock of ordinary gas dynamics) and end with $\sigma = \pi$ ($\vartheta = 0$). Here the pressure behind the shock increases with displacement along the polar from $\sigma = 1/2\pi$ toward $\sigma = \pi$. In Section 4 we saw that for $M = N$, on the other hand, the pressure behind the wave decreases with movement along the polar from $\sigma = 1/2\pi$ to $\sigma = \pi$. It can be shown that for a certain number $M = M_d$, where $1 < M_d < N$, the pressure behind the shock along the entire polar is constant and equal to the pressure in the usual normal shock. For smaller Mach numbers the pressure behind the shock increases along the polar (as σ increases from $1/2\pi$ to π), and for larger ones it decreases. For $M = N$ the weak shock waves of the second family appear, whose polar is represented by a circle. Beginning with $M = N$ and up to $M = M_c$ (see Section 5), the branches of the shock polar nowhere intersect. For $M = M_c$ their points corresponding to $\sigma = \pi$, $\vartheta = \pi$ join together. For still greater Mach numbers the juncture point of the shock polars moves toward smaller angles ϑ (the maximum angle through which the shock wave turns the stream decreases). The polar gradually deforms, tending for $M \geqslant M_b$ to one similar to the polar of ordinary gasdynamics.

We note that for $N \leqslant M \leqslant M_c$ the pressure and entropy loss behind the shock increase along the branch beginning with the weak shock wave ($\sigma = \sigma_0$). Along the second branch, moving from the normal shock toward the wave with $\sigma = \pi$, the pressure falls but the entropy loss increases. Therefore after the joining of the two branches into one polar (for

$M > M_c$) the pressure behind the shock continually increases (in moving from $\sigma = \sigma_0$ to the maximum angle of deflection of the velocity vector and then to the normal shock), but the entropy loss has a maximum at the point of juncture of the polar.

Similar to the velocity polars, the magnetic field polars can be constructed. The characteristic property of these polars is that the angle formed by the x-axis (along which the vector H_1 is directed) and the line passing through the extremities of the vectors H_1 and H_2 is equal to the angle of inclination σ of the wave.

Thus an analysis of the behavior of the shock polar of magneto-hydrodynamic shock waves has shown a number of interesting properties, which have no analog in ordinary gas dynamics. The existence of these properties opens new possibilities for practical applications.

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